Due Thurs

3.2 - Nonlinear Models

Logistic growth
$\frac{dP}{dt} = P(a - bP)$ (for large populations or when environmental factors inhibit growth)

Note: $-bP^2$ $(b>0)$ is called an <u>inhibition</u> or <u>competition</u> term.

Justification:

$\frac{dP/dt}{P} = k$ constant relative growth rate (3.1)

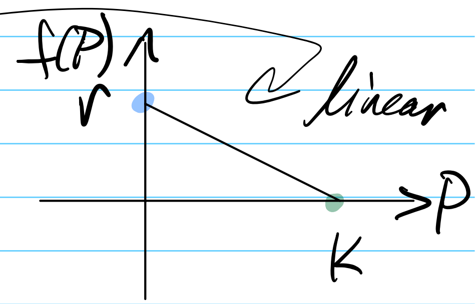
$\frac{dP/dt}{P} = f(P)$ - variable growth rate but still depends on the population

If K is the carrying capacity for the population (max pop)

$f(K) = 0$. If we let $f(0) = r$, then

in the simplest case,

$f(P) = r - \frac{r}{K}P$



linear

Then $\frac{dP}{dt} = P\left(r - \frac{r}{K}P\right)$

Relabeling gives $\frac{dP}{dt} = P(a - bP)$ Note: $K = \frac{a}{b}$

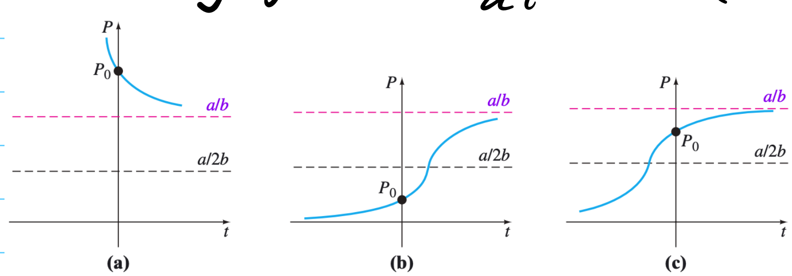


FIGURE 3.2.2 Logistic curves for different initial conditions

$P = \frac{a}{b}$ is the carrying capacity

Modified logistic growth

$$\frac{dP}{dt} = P(a - bP) \pm h$$

rate of restocking

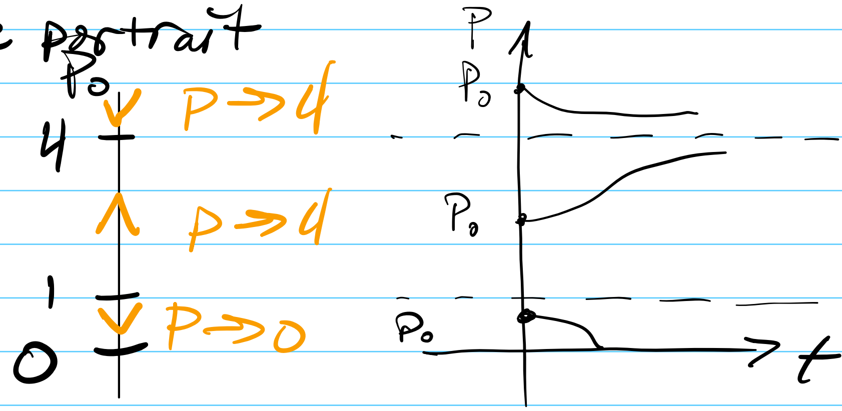
rate of harvesting

Ex: (a) If a constant number h of fish are harvested from a fishery per unit time, then a model for the population $P(t)$ of the fishery at time t is given by $\boxed{\frac{dP}{dt} = P(a - bP) - h}$, $P(0) = P_0$, where a, b, h, and P_0 are positive constants. Suppose $a = 5$, $b = 1$, and $h = 4$. Since the DE is autonomous, use the phase portrait concept of Section 2.1 to sketch representative solution curves corresponding to the cases $P_0 > 4$, $1 < P_0 < 4$, and $0 < P_0 < 1$. Determine the long-term behavior of the population in each case.

$$\frac{dP}{dt} = P(5-P) - 4 = -P^2 + 5P - 4$$
$$= -(P^2 - 5P + 4) = -(P-4)(P-1)$$

critical numbers : $P = 1, 4$

Phase portrait



$P=4$ is the carrying capacity

(b) Solve the IVP in part (a). Verify the results of your phase portrait in part (a) by using a graphing utility to plot the graph of $P(t)$ with an initial condition taken from each of the three intervals given.

$P(0) = P_0$

$$\frac{dP}{dt} = -(P-4)(P-1)$$

$$\frac{dP}{-(P-4)(P-1)} = dt$$

$$-\frac{1}{(P-4)(P-1)} = \frac{A}{P-4} + \frac{B}{P-1}$$

$$-1 = A(P-1) + B(P-4)$$

$P=4: \; A = -\frac{1}{3}$

$P=1: \; B = \frac{1}{3}$

$$\int \left(\frac{1/3}{P-1} - \frac{1/3}{P-4} \right) dP = \int dt$$

$$\frac{1}{3} \ln \left| \frac{P-1}{P-4} \right| = t + C_1$$

$C_2 = 3C_1$

$$\ln \left| \frac{P-1}{P-4} \right| = 3t + C_2$$

$C = e^{C_2}$

$$\frac{P-1}{P-4} = e^{3t + C_2} = e^{3t} e^{C_2} = C e^{3t}$$

$$P(0) = P_0 \implies \boxed{C = \frac{P_0 - 1}{P_0 - 4}}$$

For now:

$$\frac{P-1}{P-4} = \frac{P_0 - 1}{P_0 - 4} e^{3t} \qquad \frac{P-1}{P-4} = C e^{3t}$$

$$P - 1 = (P-4) C e^{3t}$$

$$P - 1 = PCe^{3t} - 4Ce^{3t}$$

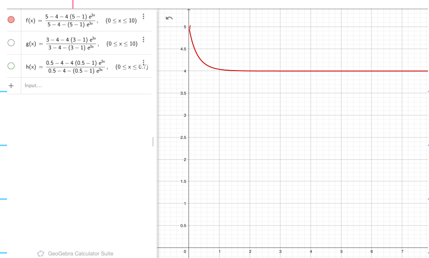
$$P(1 - Ce^{3t}) = 1 - 4Ce^{3t}$$

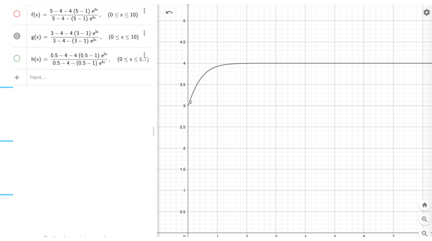
$$P = \frac{1 - 4\boxed{C}e^{3t}}{1 - \boxed{C}e^{3t}} = \frac{\left(1 - 4\frac{P_0 - 1}{P_0 - 4} e^{3t}\right)(P_0 - 4)}{\left(1 - \frac{P_0 - 1}{P_0 - 4} e^{3t}\right)(P_0 - 4)}$$

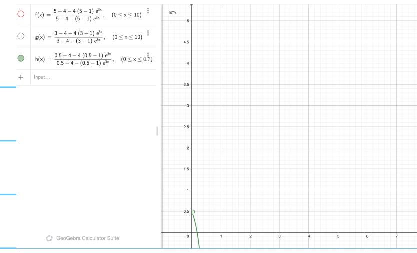
$$\boxed{P(t) = \frac{P_0 - 4 - 4(P_0 - 1)e^{3t}}{P_0 - 4 - (P_0 - 1)e^{3t}} \text{ or } = \frac{4(P_0 - 1) - (P_0 - 4)e^{-3t}}{P_0 - 1 - (P_0 - 4)e^{-3t}}}$$

(c) Use the information in parts (a) and (b) to determine whether the fishery population becomes extinct in finite time. If so, find that time.

$\rightarrow P = 0$







$p=0 \Rightarrow$ numerator $=0$

we find $\boxed{t = -\frac{1}{3}\ln\left[\frac{4(P_0-1)}{P_0-4}\right]}$

Second-order chemical reaction

$\frac{dX}{dt} = k(\alpha - X)(\beta - X)$, where $\alpha = \frac{a(M+N)}{M}$ and $\beta = \frac{b(M+N)}{N}$

a: # grams of chemical A

b: # grams of chemical B

$X(t)$ is grams of C, formed from M parts of A and N parts of B

Justification: Amts remaining at time t are $a - \frac{M}{M+N}X$ and $b - \frac{N}{M+N}X$

$\frac{dX}{dt} = k(\text{amt chem } A)(\text{amt chem } B)$

Factoring $\frac{M}{M+N}$ and $\frac{N}{M+N}$ gives the form $\frac{dX}{dt} = k(\alpha - X)(\beta - X)$

Ex: Two chemicals A and B are combined to form a chemical C. The rate, or velocity, of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 80 grams of A and 50 grams of B, and for each gram of B, 2 grams of A is used. It is observed that 10 grams of C is formed in 10 minutes. How much is formed in 40 minutes? What is the limiting amount of C after a long time? How much of chemicals A and B remains after a long time?

$a=80, \quad b=50, \quad M=2, \quad N=1$

$$\alpha = \frac{a(M+N)}{M} \qquad \beta = \frac{b(M+N)}{N}$$

$$\alpha = \frac{80(2+1)}{2} = 120, \quad \beta = \frac{50(2+1)}{1} = 150$$

$$\frac{dX}{dt} = k_1(120-X)(150-X)$$

$$\frac{dX}{(120-X)(150-X)} = k_1 dt$$

$$\frac{1}{(120-X)(150-X)} = \frac{A}{120-X} + \frac{B}{150-X}$$

$$1 = A(150-X) + B(120-X)$$

$$X=120: \ A = \frac{1}{30}, \quad X=150: \ B = -\frac{1}{30}$$

$$\int \left(\frac{1/30}{120-X} - \frac{1/30}{150-X}\right) dX = k_1 dt$$

$$\frac{1}{30} \ln\left|\frac{150-X}{120-X}\right| = k_1 t + C_1$$

Assume $X(0) = 0$

$$\ln\left|\frac{150-X}{120-X}\right| = kt + C$$

$C = \ln 5/4$ $\qquad e^{kt+C} = e^C e^{kt}$

$$\frac{150-X}{120-X} = \frac{5}{4}e^{kt} \qquad k = \frac{1}{10}\ln\frac{56}{55}$$

We know $X(10) = 10 \ldots$ $\qquad k \simeq 0.0018$

$$\boxed{X(t) = \frac{600\left(e^{0.0018t}-1\right)}{5e^{0.0018t}-4}}$$

OR $\quad X(t) = \dfrac{600\left(1-e^{-0.0018t}\right)}{5-4e^{-0.0018t}}$

As $t \to \infty$, $X \to 120\,g$

We started with 80 g of A and 50 g of B

2 g of A are used for every g of B

$120 = 80 + 40$

$80\,g$ A, $40\,g$ B

All of A is used; 10 g B are left

A was the limiting factor.

$X(40) \approx 4.2\,g$